Propagation of perturbations in non-linear spin-2 theories

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In this communication I analyze the problem of complete exceptionality of wave propagation in a class of spin 2 field theories. I show that, under the imposition of the good weak-field limit, only two Lagrangians are completely exceptional. These are the linear Fierz Lagrangian, and a Born-Infeld-like Lagrangian. As a byproduct, I reobtain the result that in a nonlinear theory, spin 2 particles follow an effective metric that depends on the nonlinearities of the Lagrangian.

It is a pleasure for me to contribute to this volume to celebrate Mario Novello's 60th birthday. Amongst the several areas of research in General Relativity (GR), Cosmology, and Field Theory that Mario has contributed to, the physics of non-linear spin 2 field theories has remained, I believe, quite unexplored. This is an area that deserves some attention: although GR has passed many observational tests with flying colours, all of them are carried out in the weak-field limit. In other words, there are no tests of the behaviour of gravity in the strong field regime. And it is precisely in this regime where the non-linearities are expected to be important. Lacking any experimental evidence, it would be important to restrict the theory space by some other means. If we start from a *generic* theory of gravitation and impose *plausible* assumptions, can we find a *class* of theories that agree with GR in the situations where the latter has been tested but with a different behavior in strong-field situations? To put it another way, how "general" is GR?

A possible way to start to build an answer to this question would be take the most general gravitational Lagrangian $\mathcal{L}$, and in a first step, restrict its form as much as possible by the imposition of sound physical principles. A second step would be to confront the predictions obtained from this Lagrangian with the available observations. We could use the standard geometrical approach to gravitation ("Einstein representation"), but a good amount of work in the "squeezing the theory space" has already been done in Electromagnetism [1, 2]. We can benefit from this work if we use the Fierz-Pauli (FP) formulation of spin 2 theories [3, 4]. In this contribution I will examine the propagation of waves in non-linear spin 2 field theories using the FP representation. In particular, I will show that out of a certain class of spin 2 nonlinear theories, only two Lagrangians satisfy the physical requirement of complete exceptionality (*i.e.* absence of shocks in wave propagation). But

let me begin first by a brief summary of the FP representation.

I. FP REPRESENTATION OF SPIN 2 FIELDS

In 1939, Fierz and Pauli showed that a spin 2 field can be represented by a third rank tensor $F_{\alpha\mu\nu}$ with the properties [3]:

$$F_{\alpha\mu\nu} + F_{\mu\alpha\nu} = 0,$$

$$F_{\alpha\mu\nu} + F_{\mu\nu\alpha} + F_{\nu\alpha\mu} = 0.$$

The dual of the object F is defined by:

$$\tilde{F}^{\alpha\mu}{}_{\lambda} \equiv \frac{1}{2}\eta^{\alpha\mu}{}_{\nu\sigma}\, F^{\nu\sigma}{}_{\lambda}.$$

We shall impose the constraint

$$\tilde{F}^{\alpha(\mu\nu)}{}_{,\alpha} = 0,$$

which guarantees that $F_{\mu\nu\alpha}$ represents a spin-two field [4] (the comma means derivative in the flat background). It also allows the introduction of a "potential" $\varphi_{\mu\nu}$:

$$F_{\alpha\mu\nu} = \frac{1}{2}\left(\varphi_{\nu[\alpha,\mu]} + \varphi_{[,\alpha}\eta_{\mu]\nu} + \eta_{\nu[\alpha}\varphi^{\lambda}{}_{\mu],\lambda}\right).$$

Taking the trace, we get

$$F_{\alpha} = \varphi_{,\alpha} - \varphi_{\alpha}{}^{\lambda}{}_{,\lambda}.$$

$F_{\mu\nu\alpha}$ satisfies

$$F^{\alpha}{}_{(\mu\nu),\alpha} \equiv -\,G^{L}{}_{\mu\nu},$$

where

$$G^{L}{}_{\mu\nu} \equiv \Box\,\varphi_{\mu\nu} - \varphi^{\epsilon}{}_{(\mu,\nu)\,,\epsilon} + \varphi_{,\mu\nu} - \eta_{\mu\nu}\,\left(\Box\varphi - \varphi^{\alpha\beta}{}_{,\alpha\beta}\right)$$

is the linearized Einstein tensor.

In the weak-field limit, we expect $\mathcal{L}$ to agree with linearized GR, so in this regime the EOM for the Fierz-Pauli tensor must be

$$F^{\alpha}{}_{(\mu\nu),\alpha} = 0,$$

which is obtainable from the Lagrangian

$$\mathcal{L} = x - y,$$

where

$$x \equiv F_{\alpha\mu\nu}\, F^{\alpha\mu\nu}, \qquad\qquad y \equiv F_{\mu}\, F^{\mu}.$$

We can define yet another invariant:

$$w \equiv F_{\alpha\beta\lambda}\tilde{F}^{\alpha\beta\lambda} = \frac{1}{2}\, F_{\alpha\beta\lambda}\, F^{\mu\nu\lambda}\, \eta^{\alpha\beta}{}_{\mu\nu},$$

which is not a topological invariant if the Lagrangian is nonlinear. In what follows, we shall consider Lagrangians of the form

$$\mathcal{L} = \mathcal{L}(z, w^2)$$

(with $z = x - y$), as suggested by the weak-field limit and to ensure parity invariance. The EOM that follow from this Lagrangian are

$$\left[\mathcal{L}_z F^{\alpha(\delta\chi)} + 2w\tilde{F}^{\alpha(\delta\chi)}\right]_{;\alpha} - 2\,\gamma^{\delta\chi}(w\mathcal{L}_{w^2}\tilde{F}^{\alpha})_{;\alpha} + +2(w\mathcal{L}_{w^2}\tilde{F}^{(\delta}\ )_{;\lambda}\,\gamma^{\chi)\lambda} = 0, \qquad (1)$$

where $\tilde{F}^{\alpha} \equiv \gamma_{\mu\nu}\tilde{F}^{\alpha\mu\nu}$, and $\gamma^{\delta\chi}$ is the metric of flat spacetime. We shall introduce in the next section one of the physical principles that will restrict the form of $\mathcal{L}$.

II. PROPAGATION OF THE DISCONTINUITIES

Shocks are unwanted features of certain nonlinear systems. A shock is an infinite discontinuity of the field, where by "infinite discontinuity" we actually mean a large change in the field value in a very short distance. Shocks form whenever wave fronts "pile up", as in the case of *caustics* in optics. However, these are due to external causes (like dielectric media). Here instead if there are shocks, they originate from the dynamics of the theory *in vacuo*. Notice also that the EOM describing the field are not valid in the region where a shock is present. We shall start then by limiting the possible Lagrangians by imposing "good propagation" (*i.e.* absence of shocks) for the spin-2 field, following the seminal work of Boillat [5] on propagation of waves in nonlinear electrodynamics. Let us begin with some conventions and notation. We shall assume that across the hypersurface S (the wave surface), given by the equation

$$\phi(x^{\alpha}) = 0, \quad \alpha = 0, 1, ..., n,$$

the highest derivatives of the dependent field variables that appear in the field equations are discontinuous. In order to state more precisely this hipothesis, it is convenient to adopt new coordinates:

$$\{x^\alpha\} \to \{\phi(x^\alpha),\ \xi^i(x^\alpha)\},\quad i = 1, 2, ..., n.$$

Our assumption is then that the jump through S of the order-q derivative (which is the highest derivative in the E.O.M) of the field component u, given by

$$\left[\frac{\partial^q u}{\partial \phi^q}\right] = \left(\frac{\partial^q u}{\partial \phi^q}\right)_{\phi=0^+} - \left(\frac{\partial^q u}{\partial \phi^q}\right)_{\phi=0^-} \equiv \delta^q u, \tag{2}$$

is finite, while $\delta^r u = 0,\ \ 0 \leq r < q$.

In order to allow for the discontinuities given in Eqn.(2), $\phi(x^\alpha)$ must be a solution of a characteristic equation of the form

$$H \equiv G^{\alpha\beta...\nu}\phi_\alpha\phi_\beta...\phi_\nu = 0 \tag{3}$$

where the completely symmetric tensor G may depend on the field and all its continuous derivatives in the case of nonlinear theories, and $\phi_\alpha \equiv \partial\phi/\partial x^\alpha$ [6].

The general theory of wave propagation shows that given an E.O.M., typically there exist several modes of propagation, and for each of them the wave surface moves with a different normal velocity. These velocities are given by $v_n^{(i)} = \lambda^{(i)}\vec{n}$, where the $\lambda^{(i)}$ are the eigenvalues of a matrix associated to the E.O.M., and $\vec{n}$ is the unit vector normal to S [6]. A *shock* (*i.e.* a jump in the field itself, which implies the divergence of its normal derivative) forms whenever wave fronts of a given propagation mode "pile up". A sufficient condition for the avoidance of shocks for a given mode is then that the velocity of the wave front be independent of the coordinate ϕ [2, 6]. That is to say,

$$\left[\partial_\phi \lambda^{(i)}\right] = 0 \tag{4}$$

When this condition is satisfied, it is said that the corresponding wave is exceptional. If this condition is satisfied by all the eigenvalues λ^i, then the system is said to be *completely exceptional* (CE). The condition for a system to be CE can be also written as [2, 5]

$$\delta H \equiv \phi_\alpha\phi_\beta...\phi_\nu\ \delta G^{\alpha\beta...\nu} = 0. \tag{5}$$

It can be shown that electromagnetic waves in Maxwell theory are CE, as well as gravitational waves in Einstein's theory [2]. Shocks are present, however, in the nonlinear electromagnetic theory described by the Euler-Heisenberg Lagrangian [7].

III. APPLICATION OF THE METHOD TO SPIN-2 NONLINEAR THEORIES

To use the sufficient condition given in the previous section, we need to determine the function H. Let's go back to the particular case of spin-2 fields with a Lagrangian

$$\mathcal{L} = \mathcal{L}(z, w^2),$$

with $z = x - y$. The EOM are

$$\left[\mathcal{L}_z F^{\alpha(\delta\chi)} + 2w\tilde{F}^{\alpha(\delta\chi)}\right]_{;\alpha} - 2\,\gamma^{\delta\chi}(w\mathcal{L}_{w^2}\tilde{F}^{\alpha})_{;\alpha} + +2(w\mathcal{L}_{w^2}\tilde{F}^{(\delta}\;)_{;\lambda}\,\gamma^{\chi)\lambda} = 0. \tag{6}$$

We shall assume that

$$\delta^{(2)}\varphi_{\alpha\beta} \equiv \pi_{\alpha\beta} \neq 0,$$

From the expression for the field F,

$$F_{\alpha\mu\nu} = \frac{1}{2}\left(\varphi_{\nu[\alpha,\mu]} + \varphi_{[,\alpha}\eta_{\mu]\nu} + \eta_{\nu[\alpha}\varphi^{\lambda}{}_{\mu],\lambda}\right),$$

we see that its first derivative is discontinuous.

The following definitions will be useful in what follows:

$$V_{\mu\nu} = \phi^{\lambda}F_{\lambda\mu\nu}, \qquad M_{\delta\chi} = \phi_{\delta}F_{\chi},$$

$$Z_{\delta\chi} = \phi_{\alpha}\tilde{F}^{\alpha}_{\delta\chi}, \qquad N_{\delta\chi} = \phi_{\chi}\tilde{F}_{\delta},$$

$$U_{\alpha} = \phi_{\nu}\pi^{\nu}_{\alpha}.$$

With the help of the rule

$$\delta^{(0)}u_{;\alpha} = \phi_{\alpha}\,\delta^{(1)}u,$$

we can take the discontinuity of the E.O.M. Omitting the index (1) from now on, we get the equations

$$V^{(\delta\chi)}\,\delta\mathcal{L}_z + 2\,(w\,\delta\mathcal{L}_{w^2} + \mathcal{L}_{w^2}\,\delta w)\,\left[W^{(\delta\chi)} - (\phi.\tilde{F})\,\gamma^{(\delta\chi)} + N^{(\delta\chi)}\right] + \mathcal{L}_z\,\phi_{\alpha}\,\delta F^{\alpha(\delta\chi)} = 0. \tag{7}$$

The variations in this equation are given by

$$\delta\mathcal{L}_z = \delta\mathcal{L}_{zz}\delta z + 2w\delta\mathcal{L}_{zw^2}\delta w,$$

(and a similar equation for $\delta\mathcal{L}_{w^2}$),

$$\delta z = -2V.\pi,$$

$$\delta w = 2(W.\pi - \pi(\phi.\tilde{F}) + U.\tilde{F}),$$

$$\phi_\alpha \delta F^{\alpha(\delta\chi)} = \phi^{(\delta}U^{\chi)} - \phi^2\ \pi^{\delta\chi} +\ \gamma^{\delta\chi}\ (\pi\phi^2 - \phi.U) - \pi\phi^\delta\phi^\chi.$$

From Eqn.(7) we conclude that

$$\pi^{\delta\chi} = aV^{(\delta\chi)} + bW^{(\delta\chi)} + c\gamma^{\delta\chi} + dN^{(\delta\chi)} + e\phi^\delta\phi^\chi + f\phi^{(\delta}U^{\chi)}.$$

Comparing this expression with Eqn.(7), we can get a system of algebraic equations for the coefficients, and obtainfrom it the function H. The calculations are rather clumsy, so we shall restrict here to the case

$$\mathcal{L} = \mathcal{L}(z),$$

for which $W^{(\delta\chi)} = N^{(\delta\chi)} = 0$, and so

$$\pi^{\delta\chi} = a\ V^{(\delta\chi)} + c\ \gamma^{\delta\chi} + e\ \phi^\delta\phi^\chi + f\ \phi^{(\delta}U^{\chi)}.$$

Projecting Eqn.(7) with the tensors appearing in $\pi^{\delta\chi}$, we get after some algebra the characteristic equation

$$H = \phi_\alpha\phi_\beta \left[\gamma^{\alpha\beta} - 2\,\frac{\mathcal{L}_{zz}}{\mathcal{L}_z}\left(F^\alpha F^\beta - 2F^{\alpha\rho\sigma}F^\beta{}_{\rho\sigma}\right)\right] = 0.$$

This is the "effective metric" obtained before in Novello, De Lorenci, and de Freitas [9]. It shows that gravitons do not move on the light cone, a prediction that will be tested in the near future by gravitational-wave observatories.

Notice that if $\mathcal{L}_{zz} = 0$, the velocity of the gravitational waves will coincide with that of electromagnetic waves. In analogy to what happens in nonlinear electromagnetism [8], in the more general case $\mathcal{L} = \mathcal{L}(z, w^2)$ we expect that

$$H = \phi_\alpha\phi_\beta \left[\gamma^{\alpha\beta} + A^{\alpha\beta}\right]$$

where

$$A^{\alpha\beta} = A^{\alpha\beta}(\mathcal{L}_z, \mathcal{L}_{zz}, \mathcal{L}zw^2, \mathcal{L}_{w^2}, \mathcal{L}_{w^2w^2}, F).$$

Imposing that $c_{gw} = c_{EM}$ would give another differential constraint to be satisfied by $\mathcal{L}$.

IV. COMPLETELY EXCEPTIONAL LAGRANGIANS

As we mentioned before, complete exceptionality is the property that guarantees that initial wavefronts evolve so as to prevent the emergence of shocks.

The condition for CE was that $\delta H = 0$. Taking the variation of

$$H = \phi_\alpha \phi_\beta \left[\gamma^{\alpha\beta} - 2\, \frac{\mathcal{L}_{zz}}{\mathcal{L}_z} \left(F^\alpha F^\beta - 2F^{\alpha\rho\sigma} F^\beta{}_{\rho\sigma} \right) \right] = 0,$$

we get a differential equation for the Lagrangian $\mathcal{L}(z)$ that governs the shock-free evolution:

$$3(\mathcal{L}^{''})^2 - \mathcal{L}^{'} \mathcal{L}^{'''} = 0. \tag{8}$$

There are only two solutions of this equation. These are the Fierz-Pauli Lagrangian,

$$\mathcal{L}(z) = az + b, \tag{9}$$

and a Born-Infeld-like Lagrangian,

$$\mathcal{L}(z) = \pm a \left(\sqrt{1 - \frac{z}{b}} \mp c \right), \tag{10}$$

where a, b and c are integration constants.

Only these two Lagrangians display the property of complete exceptionality, in close parallel to the case of nonlinear electromagnetism, with a Lagrangian that depends only on the invariant $F_{\mu\nu}F^{\mu\nu}$ [1].

Let me mention that the theory defined by Eqn.(10) (known as NDL theory) was studied in a series of papers by M. Novello and collaborators [9].

V. DISCUSSION

Let me summarize what was achieved here. First, we reobtained, for the case $\mathcal{L} = \mathcal{L}(z)$, the result that the velocity of gravitational waves will be in general different from c_{EM}. This prediction will be put to test by the observation of gravitational waves. Note however that it depends on the background fields, through the specific choice of $\mathcal{L}$.

Second, it was shown that in the simple case of $\mathcal{L} = \mathcal{L}(z)$, the method we suggested in the Introduction works: we started with a general Lagrangian, and the imposition of a *single* "sensible" physical requirement reduced drastically the theory space.

Third, the calculations for the more general case $\mathcal{L} = \mathcal{L}(z, w^2)$ (under way) are more involved, but due to the use of the Fierz-Pauli representation, they are formally similar to the analogous case

in Electromagnetism, and we can expect similar results. For this general case, most probably other requirements will be needed in addition to that of CE, like $c_{gw} = c_{EM}$, positivity of the energy (or energy conditions?), "duality"...

Finally, the allowed polarization states in these theories deserve a detailed analysis. If the number of states is to agree with that of GR (this assertion will be tested in the near future by gravitational-waves astronomy), then we would have another constraint to be satisfied by $\mathcal{L}$.

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